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THE STEADY MOTIONS OF A HEAVY ELLIPSOID FILLED WITH LIQUID ON A PLANE WITH FRICTION†

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The motion of a heavy uniform thin-walled ellipsoid of revolution, completely filled with an ideal incompressible liquid, performing uniform vortex motion is investigated. It is assumed that the ellipsoid is situated on a horizontal plane, from the side of which a normal reaction and a force of viscous sliding friction act on it. The equations of motion of the system, suitable both in the general case and in limiting cases of zero ellipsoid mass or zero liquid mass, are set up. Steady and periodic motions of the ellipsoid with the liquid are obtained. The conditions for uniform rotations of the ellipsoid about a vertically situated axis of symmetry to be stable are obtained. © 2003 Elsevier Science Ltd. All rights reserved.

It is well known that an absolutely solid axisymmetric ellipsoid (without a liquid) situated on a horizontal plane with sliding friction (a special case of a Chinese top) can rotate with arbitrary constant angular velocity around a vertically situated axis of symmetry, in which the rapid rotations of an oblate ellipsoid are unstable, and those of a prolate ellipsoid are stable (see [1–3]). The stability of uniform rotations of an axisymmetrical ellipsoid filled with liquid, executing uniform vortex motion, on a horizontal plane was previously investigated in the cases of absolutely smooth and absolutely rough planes [4], and also in the case of a plane with sliding friction, if the mass of the envelope is negligibly small [5]. Below we investigate the general case when the mass of the envelope cannot be neglected.

1. FORMULATION OF THE PROBLEM

Consider the motion of a heavy uniform thin-walled axisymmetrical ellipsoid, completely filled with an ideal incompressible liquid, which performs uniform vortex motion, on a horizontal plane taking viscous sliding friction into account. Obviously the centre of mass of the system and the principal central axes of inertia coincide with the centre S of the ellipsoid and its principal axes $Sx_1x_2x_3$ respectively.

Suppose m is the mass of the system ($m(1-\varepsilon)$ is the mass of the ellipsoid, $m\varepsilon$ is the mass of the liquid, and $\varepsilon \in [0, 1]$), $d_1, d_2 = d_1$ and d_3 are the semi-axes of the ellipsoid, $\delta = d_1/d_3$, gd_3 is the acceleration due to gravity, $d_3 v_i$, ω_i, Ω_i and γ_i ($i = 1, 2, 3$) are the projections of the velocity of the centre of mass of the ellipsoid, of the angular velocity, of half the vortex vector and of the unit vector of the ascending vertical, respectively, onto the Sx_i axis ($i = 1, 2, 3$), nd_3 is the value of the normal reaction, referred to the mass of the system, $\varkappa > 0$ is the coefficient of viscous sliding friction and $r = \sqrt{\delta^2(\gamma_1^2 + \gamma_2^2) + \gamma_3^2}$ (d_3r is the distance from the centre of the ellipsoid to the ellipsoid to the reference plane).

The equations of motion of the system, referred to the system of coordinates $Sx_1x_2x_3$ have the form (compare with the equations of motion of the system considered previously in [5])

$$\begin{aligned} \dot{v}_1 + \omega_2 v_3 - \omega_3 v_2 &= (n-g)\gamma_1 - \varkappa[v_1 + (\delta^2\omega_3\gamma_2 - \omega_2\gamma_3)r^{-1}] \\ \dot{v}_2 + \omega_3 v_1 - \omega_1 v_3 &= (n-g)\gamma_2 - \varkappa[v_2 + (\omega_1\gamma_3 - \delta^2\omega_3\gamma_1)r^{-1}] \\ \dot{v}_3 + \omega_1 v_2 - \omega_2 v_1 &= (n-g)\gamma_3 - \varkappa[v_3 + \delta^2(\omega_2\gamma_1 - \omega_1\gamma_2)r^{-1}] \end{aligned} \quad (1.1)$$

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$$(1-\varepsilon)\left[\frac{\delta^2+1}{3}\dot{\omega}_j+(-1)^{j+1}\frac{\delta^2-1}{3}\omega_{3-j}\omega_3\right]+$$
$$+\varepsilon\left[\frac{(\delta^2-1)^2}{5(\delta^2+1)}(\dot{\omega}_j-(-1)^{j+1}\omega_{3-j}\omega_3)+\frac{4\delta^2}{5(\delta^2+1)}(\dot{\Omega}_j-(-1)^{j+1}\Omega_{3-j}\omega_3)+(-1)^{j+1}\frac{2\delta^2}{5}\omega_{3-j}\Omega_3\right]=$$
$$=-(-1)^{j+1}(\delta^2-1)n\gamma_{3-j}\gamma_3 r^{-1}+\varkappa[(-1)^{j+1}(\delta^2 v_3\gamma_{3-j}-v_{3-j}\gamma_3)r+ \tag{1.2}$$
$$+(-1)^{j+1}\delta^4(\omega_2\gamma_1-\omega_1\gamma_2)\gamma_{3-j}-(\omega_j\gamma_3-\delta^2\omega_3\gamma_j)\gamma_3]r^{-2},\quad j=1,2$$
$$\frac{2}{3}(1-\varepsilon)\dot{\omega}_3+\varepsilon\left[\frac{2}{5}\dot{\Omega}_3+\frac{4}{5(\delta^2+1)}(\omega_1\Omega_2-\omega_2\Omega_1)\right]=$$
$$=\varkappa[(v_2\gamma_1-v_1\gamma_2)r+(\omega_1\gamma_1+\omega_2\gamma_2)\gamma_3-\delta^2\omega_3(\gamma_1^2+\gamma_2^2)]r^{-2}$$
$$\varepsilon[\dot{\Omega}_j+(-1)^{j+1}\frac{2\delta^2}{\delta^2+1}(\omega_{3-j}-\Omega_{3-j})\Omega_3-(-1)^{j+1}(\omega_3-\Omega_3)\Omega_{3-j}]=0,\quad j=1,2 \tag{1.3}$$
$$\varepsilon\left[\dot{\Omega}_3+\frac{2}{\delta^2+1}(\omega_1\Omega_2-\omega_2\Omega_1)\right]=0$$
$$\dot{\gamma}_1+\omega_2\gamma_3-\omega_3\gamma_2=0,\quad \dot{\gamma}_2+\omega_3\gamma_1-\omega_1\gamma_3=0,\quad \dot{\gamma}_3+\omega_1\gamma_2-\omega_2\gamma_1=0 \tag{1.4}$$
$$v_1\gamma_1+v_2\gamma_2+v_3\gamma_3+(\delta^2-1)(\omega_2\gamma_1-\omega_1\gamma_2)\gamma_3 r^{-1}=0 \tag{1.5}$$

Equations (1.1) and (1.2) express the theorems of the change in the momentum and angular momentum of the system, Eq. (1.3) expresses Helmholtz' theorem, Eq. (1.4) expresses the condition for the unit vector of the ascending vertical to be constant in a fixed system of coordinates, while Eq. (1.5) expresses the condition for undetached motion of the ellipsoid on the plane. System (1.1)–(1.5) is closed with respect to the variables v_i, ω_i, Ω_i, γ_i (i = 1, 2, 3) and n.

When $\varepsilon = 1$ (the mass of the ellipsoid is zero) Eqs (1.1)–(1.5) become the equations of motion of a massless ellipsoid filled with liquid, and when $\varepsilon = 0$ (the mass of the liquid is zero) Eqs (1.1)–(1.5) become the equations of motion of a hollow ellipsoid (in this case, obviously, Eqs (1.3) must be discarded). The case when $\varepsilon = 0$ is a special case of a Chinese top, while the case when $\varepsilon = 1$ was investigated in [5]. In general, system (1.1)–(1.5) allows of two first integrals (Helmholtz and geometric)

$$\Omega_1^2+\Omega_2^2+\delta^2\Omega_3^2=\text{const},\qquad \gamma_1^2+\gamma_2^2+\gamma_3^2=1 \tag{1.6}$$

by means of which one can eliminate the variables Ω_3 and γ_3. The variables v_3 and n can also be eliminated using the third equation of system (1.1) and the constraint equation (1.5).

2. STEADY AND PERIODIC MOTIONS

Equations (1.1)–(1.5) obviously allow of steady motions of the form

$$v_1=v_2=v_3=\gamma_1=\gamma_2=\omega_1=\omega_2=\Omega_1=\Omega_2=0 \tag{2.1}$$
$$\gamma_3=\pm 1,\qquad \omega_3=\omega,\qquad \Omega_3=\Omega$$

(ω and Ω are arbitrary constants; here $n = g$) and of the form

$$v_1=v_2=v_3=\gamma_3=\omega_3=\Omega_3=0,\quad \gamma_1=\sin\varphi,\quad \gamma_2=\cos\varphi \tag{2.2}$$
$$\omega_j=w\gamma_j,\qquad \Omega_j=W\gamma_j,\qquad j=1,2$$

(w, W and φ are arbitrary constants; here, as before, $n = g$). These solutions correspond to uniform rotations of the ellipsoid about a vertically situated axis of symmetry (solution (2.1)) or a vertically situated diameter of the equatorial cross-section (solution (2.2)).

It can easily be verified that Eqs (1.1)–(1.5) allow of periodic motions of the form

$$\begin{gathered} v_1=-\delta\omega\gamma_2, \quad v_2=\delta\omega\gamma_1, \quad v_3=0 \\ \gamma_1=\sin(\omega t+\varphi), \quad \gamma_2=\cos(\omega t+\varphi), \quad \gamma_3=0 \\ \omega_j=0, \quad \omega_3=\omega, \quad \Omega_j=\Omega\gamma_j, \quad \Omega_3=0; \quad j=1,2 \end{gathered} \tag{2.3}$$

(ω, Ω and φ are arbitrary constants; $n = g$). These solutions correspond to uniform rollings of the ellipsoid along a straight line; in this case the axis of symmetry of the ellipsoid is horizontal (as for solutions (2.2)).

Using the method proposed in [5] it can be shown that Eqs (1.1)–(1.5) also allow of periodic motions of the form

$$\begin{gathered} \gamma_1=\sqrt{1-\gamma^2}\sin[\tilde{\omega}t+\varphi], \quad \gamma_2=\sqrt{1-\gamma^2}\cos[\tilde{\omega}t+\varphi], \quad \gamma_3=\gamma \\ \tilde{\omega}=(1-\delta^2)\omega\gamma; \quad v_1=v_2=v_3=0 \quad (n=g) \\ \omega_j=\delta^2\omega\gamma_j, \quad \omega_3=\omega\gamma, \quad \Omega_j=\mu\delta^2\Omega\gamma_j, \quad \Omega_3=\Omega\gamma; \quad j=1,2 \\ (\mu=2\delta^2\omega[(\delta^2-1)\Omega+\delta^2(\delta^2+1)\omega]^{-1}) \end{gathered} \tag{2.4}$$

Here ω, Ω and $\gamma(|\gamma| \leqslant 1)$ are constants, connected by the relation

$$\frac{15g}{\omega^2\delta^4\sqrt{\delta^2(1-\gamma^2)+\gamma^2}}=\frac{6\varepsilon k^2+(1-\delta^2)(5-8\varepsilon)k-\delta^2[5(\delta^2+1)-2\varepsilon(\delta^2+4)]}{(1-\delta^2)k-\delta^2(\delta^2+1)}, \quad k=\frac{\Omega}{\omega} \tag{2.5}$$

where φ is an arbitrary constant. These solutions correspond to regular precessions of the ellipsoid and is transformed into uniform rotations (2.1) or (2.2) where $\gamma^2 = 1$ or $\gamma = 0$ respectively (in the second case $w = \delta^2\omega$, $W = 2k\delta^4\omega[k(\delta^2 - 1) + \delta^2(\delta^2 + 1)]^{-1}$). When $\varepsilon = 1$ relation (2.5) is identical with the corresponding relation obtained previously in [5], and when $\varepsilon = 0$ it takes the form

$$3g=\omega^2\delta^4\sqrt{\delta^2(1-\gamma^2)+\gamma^2}$$

and defines the relation between the constants ω and γ in the regular precessions of a hollow ellipsoid on a horizontal plane with friction. When $k = 1$ ($\Omega = \omega$) the right-hand side of relation (2.5) takes the form

$$\frac{(5-2\varepsilon)\delta^4+2(5-8\varepsilon)\delta^2-(5-2\varepsilon)}{\delta^4+2\delta^2-1} \tag{2.6}$$

Taking into account the fact that its left-hand side is positive, we conclude that regular precessions of an ellipsoid filled with liquid, of the form (2.4) (when $\Omega = \omega$) only exist when the quantity (2.6) is positive.

If $\delta^2 > \sqrt{2} - 1$ or $\delta^2 < \sqrt{2} - 1$, this condition is satisfied when $\delta^2 > \delta_0^2(\varepsilon)$ or $\delta^2 < \delta_0^2(\varepsilon)$ respectively, where

$$\delta_0^2(\varepsilon)=\frac{-5+8\varepsilon+\sqrt{50-100\varepsilon+68\varepsilon^2}}{5-2\varepsilon} \tag{2.7}$$

The function $\delta_0^2(\varepsilon)$ obviously increases monotonically in the section [0, 1], where

$$\delta_0^2(0)=\sqrt{2}-1, \quad \delta_0^2(5/8)=1, \quad \delta_0^2(1)=\sqrt{2}+1$$

Hence, in the case of an oblate ellipsoid ($\delta > 1$) regular precessions (2.4) ($\Omega = \omega$) exist for any oblateness ($\forall\delta > 1$), if the mass of the liquid is no greater than $5/8$ of the mass of the system ($0 \leqslant \varepsilon \leqslant 5/8$), and only in the case of considerable oblateness ($\delta > \delta_0 > 1$) in the opposite case ($1 \geqslant \varepsilon > 5/8$). In the case of a prolate ellipsoid ($\delta < 1$) regular precessions (2.4) ($\Omega = \omega$) exist for any ratio of the mass of the liquid to the mass of the system ($\forall\varepsilon \in [0, 1]$) for an extremely prolate ellipsoid ($\delta^2 < \sqrt{2} - 1$), and only provided the mass of the liquid is less than $5/8$ of the mass of the system ($0 \leqslant \varepsilon < 5/8$), for a slightly prolate ellipsoid ($1 > \delta > \delta_0$).

In particular where $\varepsilon = 0$ (a hollow ellipsoid) regular precessions exist for any ellipsoid of revolution, and when $\varepsilon = 1$ (a massless ellipsoid, filled with liquid) they only exist for a very oblate ellipsoid ($\delta^2 > \sqrt{2} + 1$) or a very prolate ellipsoid ($\delta^2 < \sqrt{2} - 1$) (compare with the results for the cases of a solid ellipsoid without a liquid [2] and a massless ellipsoid filled with liquid [5]).

3. THE STABILITY OF UNIFORM ROTATIONS OF THE ELLIPSOID ABOUT THE AXIS OF SYMMETRY

We will consider steady motion (2.1) and write linearized equations of the perturbed motion in its neighbourhood, assuming

$$\gamma_3 = 1 + \gamma_3', \quad n = g + n', \quad \omega_3 = \omega + \omega_3', \quad \Omega_3 = \Omega + \Omega_3'$$

and retaining the previous notations for the remaining variables v_i ($i = 1, 2, 3$), γ_i, ω_j and Ω_j ($j = 1, 2$). Eliminating the variables γ_3', v_3, n' and Ω_3' using relations (1.5) and (1.6) and the third equation of system (1.1), we obtain after simple but fairly lengthy calculations,

$$\dot{\gamma}_j - (-1)^{j+1}(\omega\gamma_{3-j} + \omega_{3-j}) = 0, \quad j = 1,2 \tag{3.1}$$

$$\dot{\Omega}_j - (-1)^{j+1}\left(\omega\Omega_{3-j} + \frac{\delta^2 - 1}{\delta^2 + 1}\Omega\Omega_{3-j} - \frac{2\delta^2}{\delta^2 + 1}\Omega\omega_{3-j}\right) = 0, \quad j = 1,2 \tag{3.2}$$

$$\dot{v}_j - (-1)^{j+1}\omega v_{3-j} + \varkappa v_j + (-1)^{j+1}\varkappa\omega\delta^2\gamma_{3-j} - (-1)^{j+1}\varkappa\omega_{3-j} = 0, \quad j = 1,2 \tag{3.3}$$

$$\begin{aligned}
&\frac{5}{3}\frac{\delta^2+1}{\delta^2-1}(1-\varepsilon)\left[\frac{\delta^2+1}{\delta^2-1}\dot{\omega}_j + (-1)^{j+1}\omega\omega_{3-j}\right] + \varepsilon\dot{\omega}_j - (-1)^{j+1}\varepsilon\omega\omega_{3-j} + \\
&+5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}\omega_j + (-1)^{j+1}\frac{2\delta^2}{\delta^2+1}\varepsilon\Omega\omega_{3-j} - 5\varkappa\frac{\delta^2(\delta^2+1)}{(\delta^2-1)^2}\omega\gamma_j + \\
&+(-1)^{j+1}\left[5g\frac{\delta^2+1}{\delta^2-1}\gamma_{3-j} + \frac{4\delta^2}{\delta^4-1}\varepsilon\Omega\Omega_{3-j} + 5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}v_{3-j}\right] = 0, \quad j = 1,2
\end{aligned} \tag{3.4}$$

$$\dot{\omega}_3' = 0 \tag{3.5}$$

Equation (3.5) is clearly separated from system (3.1)–(3.4), and the zero root of the characteristic equation for system (3.1)–(3.5), related to the arbitrariness of the parameter ω in solution (2.1), corresponds to this equation. Hence, the stability of this solution depends on the roots of the characteristic equation for system (3.1)–(3.4).

Assuming

$$x = (\gamma_1 + i\gamma_2)e^{i\omega t}, \quad y = (\Omega_1 + i\Omega_2)e^{i\omega t}$$
$$z = (\omega_1 + i\omega_2)e^{i\omega t}, \quad v = (v_1 + iv_2)e^{i\omega t}$$

we can reduce the eight-order system (3.1)–(3.4) in terms of eight real variables to a fourth-order system in four complex variables x, y, z and v

$$\dot{x} - iz = 0$$
$$\dot{y} + i\frac{\delta^2 - 1}{\delta^2 + 1}\Omega y - 2i\frac{\delta^2}{\delta^2 + 1}\Omega z = 0$$
$$\dot{v} + \varkappa v - i\varkappa\omega\delta^2 x + i\varkappa z = 0$$

$$\left[\delta^2+1-\frac{2}{5}\varepsilon\frac{\delta^4+8\delta^2+1}{\delta^2+1}\right]\dot{z}+\left[3\varkappa-2i\delta^2\left(\frac{3}{5}\varepsilon\frac{(\delta^2-1)^2}{(\delta^2+1)^2}\Omega+(1-\varepsilon)\omega\right)\right]z-$$
$$-3[\varkappa\omega\delta^2+i(\delta^2-1)g]x-\frac{12}{5}i\varepsilon\frac{\delta^2(\delta^2-1)}{(\delta^2+1)^2}\Omega y-3i\varkappa v=0 \tag{3.6}$$

Suppose $\Omega = \omega$. Then the characteristic equation of system (3.6) has the form

$$f(\lambda)=p_0\lambda^4+(p_1+iq_1)\lambda^3+(p_2+iq_2)\lambda^2+(p_3+iq_3)\lambda+iq_4=0 \tag{3.7}$$

$$p_k=a_k+\alpha_k\varepsilon,\quad k=0,1,2,3;\qquad q_j=b_j+\beta_j\varepsilon,\quad q_{j+2}=b_{j+2};\quad j=1,2$$

$$a_0=-\frac{b_1}{\omega}=\frac{(\delta^2+1)^2}{3(\delta^2-1)^2},\qquad a_1=\varkappa\frac{(\delta^2+1)(\delta^2+4)}{3(\delta^2-1)^2}$$

$$a_2=\frac{\delta^2+1}{\delta^2-1}g+\frac{2}{3}\frac{\delta^2}{\delta^2-1}\omega^2,\qquad a_3=\varkappa\left[\frac{\delta^2+1}{\delta^2-1}g+\frac{5}{3}\frac{\delta^2}{\delta^2-1}\omega^2\right]$$

$$b_2=-\frac{2\varkappa\omega(2\delta^4+\delta^2+2)}{3(\delta^2-1)^2},\qquad b_3=\omega g,\quad b_4=\omega g\varkappa$$

$$\alpha_0=\frac{\alpha_1}{\varkappa}=-\frac{\beta_1}{\varkappa}=-\frac{\beta_2}{\omega\varkappa}=-\frac{2}{15}\frac{\delta^4+8\delta^2+1}{(\delta^2-1)^2}$$

$$\alpha_2=\frac{\alpha_3}{\varkappa}=-\frac{4}{15}\frac{\delta^2}{\delta^2-1}\omega^2$$

The permanent rotations (2.1) ($\Omega = \omega$) are obviously stable and also are asymptotically stable with respect to all the variables, apart, generally speaking, from the variables Ω_3 and ω_3, if all the roots of Eq. (3.7) lie in the left half-plane, and unstable if at least one root lies in the right half-plane.

All the roots of Eq. (3.7) have negative real parts if and only if the matrix

$$\left\|\begin{matrix} p_0 & -q_1 & -p_2 & q_3 & 0 & 0 & 0\\ 0 & p_0 & -q_1 & -p_2 & q_3 & 0 & 0\\ 0 & 0 & p_0 & -q_1 & -p_2 & q_3 & 0\\ 0 & 0 & 0 & p_1 & -q_2 & -p_3 & q_4\\ 0 & 0 & p_1 & -p_2 & -p_3 & q_4 & 0\\ 0 & p_1 & -q_2 & -p_3 & -q_4 & 0 & 0\\ p_1 & -q_2 & -p_3 & -q_4 & 0 & 0 & 0 \end{matrix}\right\| \tag{3.8}$$

is innerly positive [6]. The conditions for matrix (3.8) to be innerly positive have the form

$$(\delta^2-1)\{15(\delta^2+1)^2[5(\delta^4+5\delta^2+4)-2(\delta^4+8\delta^2+1)\varepsilon]g-$$
$$-\delta^4\omega^2[125(\delta^2+1)^3-10(7\delta^6+63\delta^4+75\delta^2+55)\varepsilon+8(\delta^6+15\delta^4+57\delta^2+7)\varepsilon^2]\}>0 \tag{3.9}$$

$$(\delta^2-1)\{1125(\delta^2+1)^5g^3-75(\delta^2+1)^2\delta^4g^2\omega^2[5(\delta^2+1)^3-2(\delta^6+9\delta^4+9\delta^2+37)\varepsilon]-$$
$$-120\delta^4g\omega^4[5(\delta^4+2\delta^2-1)(2\delta^4+\delta^2-4)-2(2\delta^8+17\delta^6+5\delta^4+3\delta^2+1)\varepsilon]\varepsilon-$$
$$-8\delta^8\omega^6[125(\delta^4+2\delta^2-1)-10(7\delta^4+44\delta^2-7)\varepsilon+8(\delta^4+8\delta^2-1)\varepsilon^2]\varepsilon\}>0 \tag{3.10}$$

$$(\delta^2-1)\{15(\delta^4+2\delta^2-1)g-\delta^4\omega^2[5(\delta^4+2\delta^2-1)-2(\delta^4+8\delta^2-1)\varepsilon]\}\varepsilon>0 \tag{3.11}$$

Hence, when inequalities (3.9)–(3.11) are satisfied, the uniform rotations (2.1) are stable, and also asymptotically stable with respect to a part of the variables. If at least one of these inequalities is strictly violated, the uniform rotations (2.1) are unstable.

4. ANALYSIS OF THE RESULTS

An investigation of inequalities (3.9)–(3.11) shows that the stability of uniform rotations (2.1) ($\Omega = \omega$) depends considerably on the parameter $\varepsilon \in [0, 1]$, characterizing the ratio of the mass of the liquid to the mass of the whole system.

If the liquid is fairly heavy ($\varepsilon \geq 5/8$), uniform rotations of a prolate ellipsoid ($\delta < 1$) are unstable for any angular velocity, while uniform rotations of a slightly oblate ellipsoid ($1 < \delta \leq \delta_0(\varepsilon)$) are stable for any angular velocity; uniform rotations of an extremely oblate ellipsoid ($\delta > \delta_0(\varepsilon)$) are stable for low angular velocities ($\omega^2 < \omega_0^2(\varepsilon)$) and unstable for high angular velocities ($\omega^2 > \omega_0^2(\varepsilon)$). Here $\delta_0(\varepsilon)$ is given by formula (2.7) while

$$\omega_0^2(\varepsilon) = \frac{3g}{\delta^4} \frac{5(\delta^4 + 2\delta^2 - 1)}{5(\delta^4 + 2\delta^2 - 1) - 2(\delta^4 + 8\delta^2 - 1)\varepsilon} \tag{4.1}$$

If the liquid is fairly light ($0 < \varepsilon < 5/8$), uniform rotations of an oblate ellipsoid ($\delta > 1$) are stable or unstable for low or high angular velocities ($\omega^2 < \omega_0^2(\varepsilon)$ or $\omega^2 > \omega_0^2(\varepsilon)$) respectively, uniform rotations of a slightly prolate ellipsoid ($\delta_0(\varepsilon) < \delta < 1$) are stable or unstable for high or low angular velocities ($\omega^2 > \omega_0^2(\varepsilon)$ or $\omega^2 < \omega_0^2(\varepsilon)$) respectively, while uniform rotations of an extremely prolate ellipsoid ($\delta < \delta_0(\varepsilon) < 1$) are unstable for any angular velocity. Obviously, a change in the stability of uniform rotations (2.1) ($\Omega = \omega$) for critical values of the angular velocity (4.1) is related to the onset of regular precessions (2.4) ($\Omega = \omega$) of the ellipsoid (Andronov–Hopf bifurcation [7]).

In Fig. 1 ($5/8 \leq \varepsilon < 1$) and Fig. 2 ($0 < \varepsilon < 5/8$) we show regions of stability (hatched) and instability of uniform rotations (2.1) ($\Omega = \omega$) as a function of the degree of oblateness (prolateness) of the ellipsoid and of its angular velocity. Note that the vertical asymptote $\delta = \delta_0(\varepsilon)$ of the curvilinear boundary of the stability region, specified by formula (4.1), approaches the straight line $\delta = \sqrt{2} + 1$ as $\varepsilon \to 1 - 0$, $\delta = 1$ as $\varepsilon \to 5/8$ and $\delta = \sqrt{2} - 1$ as $\varepsilon \to +0$. Hence, the region of stability, indicated in Fig. 1 ($5/8 \leq \varepsilon < 1$), transfers continuously as $\varepsilon \to 1 - 0$ into the region of stability [5] of uniform rotations of a massless ellipsoid ($\varepsilon = 1$) filled with a liquid.

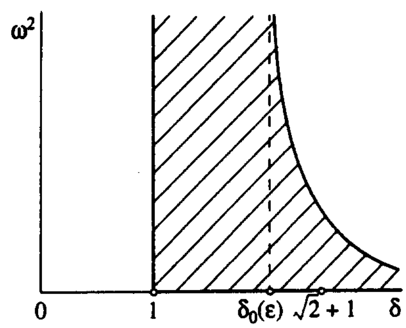

Fig. 1

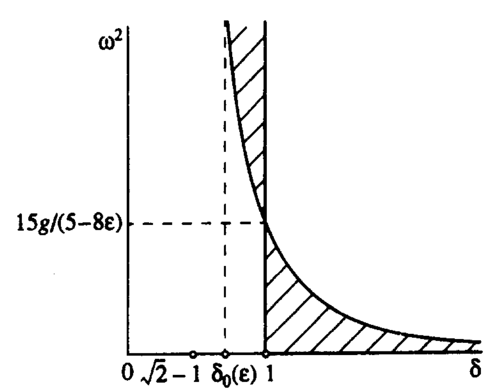

Fig. 2

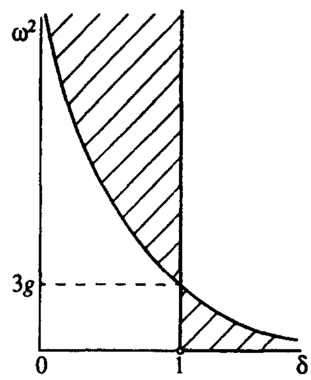

Fig. 3

As $\varepsilon \to +0$ no such continuous transfer is observed: in Fig. 3 we show the region of stability of uniform rotations of a hollow ellipsoid (containing no liquid); the curvilinear boundary of this region is defined by the relation $\omega^2 = 3g/\delta^4$ [1–3]. Hence, the presence of even a light liquid ($0 < \varepsilon \ll 1$) somewhat expands the region of stability of uniform rotations (2.1) ($\Omega = \omega$) of an oblate ellipsoid and sharply narrows the region of stability of these rotations of a prolate ellipsoid (compare Figs 2 and 3). This difference in the stability regions when $\varepsilon = +0$ (there is a liquid in the cavity but it is very light) and $\varepsilon = 0$ (there is no liquid in the cavity) is completely explicable. In the second case ($\varepsilon = 0$) we must discard Eqs (1.3) and of course also Eqs (3.2), the second equation of system (3.6), etc.

Note also that this difference can also be explained strictly mathematically. When $\varepsilon = 0$ the left-hand side of inequality (3.11) vanishes. Consequently, the characteristic equation (3.7), which can be represented in the form

$$f_0(\lambda) + \varepsilon f_1(\lambda) = 0 \tag{4.2}$$

$$f_0 = a_0\lambda^4 + (a_1 + ib_i)\lambda^3 + (a_2 + ib_2)\lambda^2 + (a_3 + ib_3)\lambda + ib_4$$

$$f_1 = \alpha_0\lambda^4 + (\alpha_1 + i\beta_1)\lambda^3 + (\alpha_2 + i\beta_2)\lambda^2 + \alpha_3\lambda$$

has a pure imaginary root λ_0 when $\varepsilon = 0$. This root is easily obtained: $\lambda_0 = -i\omega(\delta^2 - 1)/(\delta^2 + 1)$ ($f_0(\lambda_0) = 0$). When $0 < \varepsilon \ll 1$, Eq. (4.2) has a root $\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \ldots$, where $\lambda_1 = -f_1(\lambda_0)/f_1'(\lambda_0) = \xi + i\eta$ (the prime denotes a derivative with respect to λ). Calculating λ_1, we have

$$\xi = \frac{8\delta^4\omega^6(1 - 2\delta^2 - \delta^4)}{5(\delta^2 + 1)^5(A^2 + B^2)}$$

$$A = \frac{\varkappa}{3(\delta^2 - 1)(\delta^2 + 1)}[3(\delta^2 + 1)^2 g - 2\omega^2(3\delta^4 + 4\delta^2 - 2)]$$

$$B = \frac{\omega}{3(\delta^2 + 1)}[-3(\delta^2 + 1)g + \omega^2(3\delta^2 - 1)]$$

Hence, if $\delta^2 < \sqrt{2} - 1$, then $\xi > 0$ and the root $\lambda(\varepsilon)$ has a positive real part of the order of ε. Consequently, the presence of even a light liquid in the cavity leads to Lyapunov instability of all the uniform rotations of an extremely prolate ellipsoid ($\delta^2 < \sqrt{2} - 1$). However, if the angular velocity of the ellipsoid is sufficiently high ($\omega^2 > 3g/\delta^4$), then when $0 < \varepsilon \ll 1$ all the remaining roots of Eq. (4.2) have negative real parts and, in times of the order of $1/\varepsilon$, perturbed motions of the ellipsoid are close to unperturbed motion.

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